Representation of Qubit States Using the Bloch Sphere

1. Introduction

Quantum computing extends beyond classical binary logic by using **quantum bits (qubits)**, which can exist in a **superposition** of states.
To visualize and understand these states intuitively, physicists and engineers use a geometric tool called the **Bloch Sphere**.

The **Bloch Sphere** provides a **3D geometrical representation** of a single qubit's state, capturing both its **probability amplitudes** and **phase**.
It serves as a powerful conceptual model to understand quantum state evolution, measurement, and gate operations.

2. The Concept of the Bloch Sphere

A qubit's state can be described mathematically as:

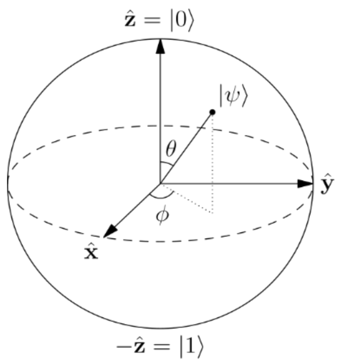
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where:

- $\alpha, \beta \in \mathbb{C}$ (complex numbers)
- $|\alpha|^2 + |\beta|^2 = 1$

This normalization ensures that the qubit lies on the **surface of a unit sphere (r = 1)** in 3D space.

The Bloch sphere visualizes this by mapping every qubit state to a point on or inside the sphere.



The **Bloch Sphere** is a **3D geometric representation of a single qubit**. It illustrates how a qubit's quantum state can be described by two angles — **θ (theta)** and **φ (phi)** — on the surface of a sphere of unit radius. Every pure state of a single qubit corresponds to a point on the **surface** of this sphere. The **north pole** and **south pole** correspond to the classical states $|0\rangle$ and $|1\rangle$ respectively.

The Mathematical State: The general state of a qubit is expressed as:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Here:

- $\theta \rightarrow$ **polar angle** (from the z-axis, between 0 and π)
- $\phi \rightarrow$ **azimuthal angle** (around the z-axis, between 0 and 2π)

Explanation of Each Component in the Figure:

Let's decode every label and line:

(a) Axes

- **ẑ-axis** → vertical axis
  - Top (north pole): $|0\rangle$
  - Bottom (south pole): $|1\rangle$
- **x̂-axis** and **ŷ-axis** → horizontal axes
  - These represent phase relationships (real and imaginary components).

So, the qubit's vector (arrow) extends from the center of the sphere to the point **$|\psi\rangle$**, determined by angles θ and φ.

(b) The State Vector $|\psi\rangle$

The **arrow labeled $|\psi\rangle$** represents the **quantum state** of the qubit. Its direction determines the amplitudes (α, β) in the state:

$|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$ and those amplitudes are derived from the coordinates given by θ and ϕ.

(c) The Angles

1. **θ (theta)** — Polar angle
   - Measured from the **z-axis** downward.
   - Determines the *probability* of measuring $|0\rangle$ or $|1\rangle$.
     - Probability($|0\rangle$) = $\cos^2(\theta/2)$
     - Probability($|1\rangle$) = $\sin^2(\theta/2)$

   ✅ Example:

   - $\theta = 0 \rightarrow |\psi\rangle = |0\rangle$ (points at north pole)
   - $\theta = \pi \rightarrow |\psi\rangle = |1\rangle$ (points at south pole)
   - $\theta = \pi/2 \rightarrow$ equal superposition (on equator)

2. **φ (phi)** — Azimuthal (phase) angle
   - Measured in the x–y plane from the x-axis.
   - Controls the **relative phase** between $|0\rangle$ and $|1\rangle$.
   - This phase determines where on the equator the state lies.

   ✅ Example:

    - $\varphi = 0$ → along x-axis → $(|0\rangle + |1\rangle)/\sqrt{2}$
    - $\varphi = \pi/2$ → along y-axis → $(|0\rangle + i|1\rangle)/\sqrt{2}$
    - $\varphi = \pi$ → opposite side → $(|0\rangle - |1\rangle)/\sqrt{2}$

(d) The Equator (Dashed Circle)

The **dashed circle** in the middle represents the **equatorial plane** of the Bloch sphere. States lying on this plane ($\theta = \pi/2$) have **equal probabilities** for $|0\rangle$ and $|1\rangle$ but different **phases** (φ determines direction).

For instance:

- On the x-axis: $|+\rangle=(|0\rangle+|1\rangle)/\sqrt{2}$
- On the –x-axis: $|-\rangle=(|0\rangle-|1\rangle)/\sqrt{2}$
- On the y-axis: $|+i\rangle=(|0\rangle+i|1\rangle)/\sqrt{2}$
- On the –y-axis: $|-i\rangle=(|0\rangle-i|1\rangle)/\sqrt{2}$

(e) Arrows and Notations

- **$\hat{z} = |0\rangle$** → indicates the north pole of the sphere (the "ground" state).
- **$-\hat{z} = |1\rangle$** → indicates the south pole (the "excited" state).
- The vector **$|\psi\rangle$** shows the actual qubit's state orientation in space.
- **θ and φ** are shown as the **geometrical angles** between axes and the qubit vector.

---

4. Physical Meaning

The Bloch sphere gives a physical picture of **how a qubit behaves:**

- **Rotation around the X-axis** → changes population between $|0\rangle$ and $|1\rangle$.
- **Rotation around the Z-axis** → changes only the phase (φ).
- **Hadamard gate (H)** → moves the state from the pole to the equator.
- **Measurement** collapses the state vector to either $|0\rangle$ or $|1\rangle$ along the z-axis.

Thus, **quantum gates correspond to rotations**, and **measurements correspond to projections**.

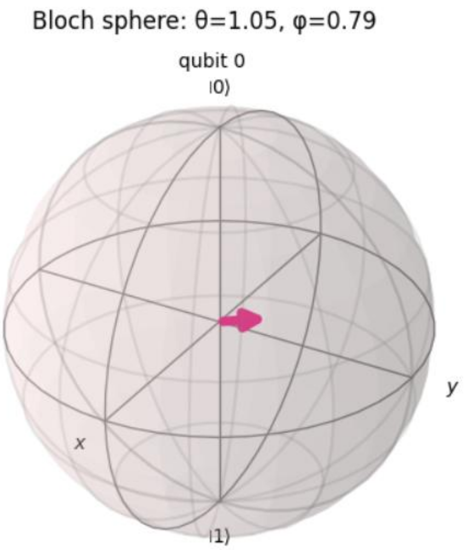
5. Example: Bloch Sphere Visualization in Qiskit

Below is the Qiskit code to visualize the same concept.

```
%matplotlib inline
import matplotlib.pyplot as plt
```

```
plt.close('all')            # clear previous figures

from qiskit import QuantumCircuit
from qiskit.quantum_info import Statevector
from qiskit.visualization import plot_bloch_multivector
import numpy as np
from IPython.display import display

# Prepare a parameterized state:
|ψ> = cos(theta/2)|0> + e^{i phi} sin(theta/2)|1>
theta = np.pi/3   # e.g. 60 degrees
phi = np.pi/4     # e.g. 45 degrees

qc = QuantumCircuit(1)
qc.ry(theta, 0)   # moves down from north pole by theta
qc.rz(phi, 0)     # add phase phi

# Compute statevector
state = Statevector.from_instruction(qc)
print(f"State |ψ⟩ = cos({theta/2:.2f})|0⟩ + e^(i{phi:.2f}) sin({theta/2:.2f}) |1⟩")
print(state)      # show amplitudes

# plot_bloch_multivector RETURNS a Figure. Capture + display it.
fig = plot_bloch_multivector(state)
fig.suptitle(f"Bloch sphere: θ={theta:.2f}, ϕ={phi:.2f}", fontsize=12)
display(fig)
```

```
State |ψ⟩ = cos(0.52)|0⟩ + e^(i0.79) sin(0.52) |1⟩
Statevector([0.80010315-0.33141357j, 0.46193977+0.19134172j],
            dims=(2,))
```



Observation:

- The Bloch vector appears at angle θ from the z-axis and φ from the x-axis — exactly as shown in the figure.

⚡ 6. Real-World Analogy

Imagine the Bloch sphere like the **Earth globe**:

- North Pole → |0⟩
- South Pole → |1⟩
- Equator → balanced mix of both (superposition)
- Longitude (φ) → phase of the wavefunction
- Latitude (θ) → probability bias toward |0⟩ or |1⟩

Thus, any qubit state is like a point on Earth defined by (θ, φ).

7. Applications of the Bloch Sphere Concept

1. **Quantum Algorithm Design:**
   Helps visualize gate sequences and how they rotate the qubit state.
2. **Quantum Error Correction:**
   Errors like dephasing or relaxation appear as vector shrinkage toward the center (loss of coherence).
3. **Quantum Hardware Calibration:**
   Engineers use Bloch vectors to check accurate π/2 and π rotations.
4. **Quantum Communication:**
   Photon polarization states in QKD (Quantum Key Distribution) directly map to points on the Bloch sphere.
5. **Quantum Learning Models:**
   Bloch vectors are used as inputs/outputs in Quantum Neural Networks and classification tasks.

Summary in Simple Words

- The **Bloch sphere** is a 3D map of all possible single-qubit states.
- **|0⟩ and |1⟩** are poles; every other point is a **superposition**.
- Angles θ and φ completely define a qubit’s quantum state.
- **Quantum gates** = **rotations** on the sphere.
- **Measurements** = **collapsing** the qubit to a pole.

Representing QuBits states using Bloch Sphere

- The state of a qubit can be represented as a point on the Bloch Sphere
- It is a unit sphere, where r=1.

Qubit State Representation

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where α and β are complex numbers.

Expressing α and β:

$$\alpha = |\alpha|e^{i\phi_\alpha}, \quad \beta = |\beta|e^{i\phi_\beta}$$

The state can also be written as:

$$|\psi\rangle = |\alpha|e^{i\phi_\alpha}|0\rangle + |\beta|e^{i\phi_\beta}|1\rangle$$

This can be factored as:

$$|\psi\rangle = e^{i\phi_\alpha}\left[|\alpha||0\rangle + |\beta|e^{i(\phi_\beta - \phi_\alpha)}|1\rangle\right]$$

Here, $\phi = \phi_\beta - \phi_\alpha$ is the **relative phase**.

Since the global phase $e^{i\phi_\alpha}$ is insignificant, it can be ignored:

$$|\psi\rangle = |\alpha||0\rangle + |\beta|e^{i\phi}|1\rangle$$

Set:

$$|\alpha| = \cos(\theta/2), \quad |\beta| = \sin(\theta/2)$$

Then:

$$\cos^2(\theta/2) + \sin^2(\theta/2) = 1$$

So, the qubit state can be written as:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

Magnitudes and Trigonometric Representation:

We know:

$$|\alpha|^2 + |\beta|^2 = 1$$

Set:

$$|\alpha| = \cos(\theta/2), \quad |\beta| = \sin(\theta/2)$$

Then:

$$\cos^2(\theta/2) + \sin^2(\theta/2) = 1$$

So, the qubit state can be written as:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

- A Bloch sphere uses its three axes to represent a **qubit's** state. The state vector originates in the center of the sphere and terminates at a point with **z, x, and y** coordinates.
- The **z-axis** represents the probability of the qubit being measured as a **0 or a 1**.
- The **x-axis** represents the **real part** of the state vector.
- The **y-axis** represents the **imaginary part** of the state vector.

Qubit State in Polar Coordinates

To represent this state in polar coordinates on the Bloch sphere, we express α and β using two angles θ and ϕ:

1. θ: This is the polar angle (latitude) on the Bloch sphere, ranging from 0 to π.
2. ϕ: This is the azimuthal angle (longitude) on the Bloch sphere, ranging from 0 to 2π.

Given these angles, the qubit state $|\psi\rangle$ can be written as:

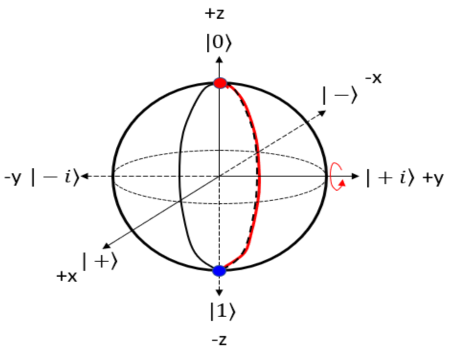
$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

- $\cos\left(\frac{\theta}{2}\right)$ corresponds to the probability amplitude for the qubit being in the $|0\rangle$ state.
- $\sin\left(\frac{\theta}{2}\right)$ corresponds to the probability amplitude for the qubit being in the $|1\rangle$ state.
- $e^{i\phi}$ introduces a phase factor for the $|1\rangle$ component.

$$\boxed{|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle}$$



$$\boxed{|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}}$$

- $\theta = 0$
- ϕ can be any value, but typically we take $\phi = 0$.

$$\boxed{|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}}$$

- $\theta = \pi$
- ϕ can be any value, but typically we take $\phi = 0$.

$$\boxed{|i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}}$$

- $\theta = \frac{\pi}{2}$ (since both $|0\rangle$ and $|1\rangle$ components have equal magnitude)
- $\phi = \frac{\pi}{2}$ (due to the i phase factor, which corresponds to a phase of $\frac{\pi}{2}$).

$$\boxed{|-i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}}$$

- $\theta = \frac{\pi}{2}$ (since both $|0\rangle$ and $|1\rangle$ components have equal magnitude)
- $\phi = -\frac{\pi}{2}$ (due to the $-i$ phase factor, which corresponds to a phase of $-\frac{\pi}{2}$).

$$\boxed{|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}$$

- $\theta = \frac{\pi}{2}$ (since both $|0\rangle$ and $|1\rangle$ components have equal magnitude)
- $\phi = 0$.

$$\boxed{|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}$$

- $\theta = \frac{\pi}{2}$ (since both $|0\rangle$ and $|1\rangle$ components have equal magnitude)
- $\phi = \pi$ (due to the -1 phase factor).