# Mathematical Model for Quantum Computing

Quantum computing relies on mathematical frameworks that extend beyond classical linear algebra and complex arithmetic. While classical computing manipulates binary states (0 or 1), quantum computing deals with **quantum states**—vectors in a **complex vector space**—that can exist in superpositions of both 0 and 1 simultaneously.

To understand the working principles of qubits and quantum operations, we must first establish the mathematical foundation that supports quantum theory. This includes complex numbers, vector spaces, matrices, operators, and the structure known as **Hilbert space**, which forms the core mathematical environment for quantum mechanics.

## 1. Complex Numbers

In classical mathematics, we deal primarily with **real numbers**, which are points on a one-dimensional number line. In quantum mechanics and quantum computing, we require **complex numbers**, which extend this system to two dimensions.

A **complex number** z is written as:

$$z = a + bi$$

where:

- a → **real part**
- b → **imaginary part**
- i → imaginary unit (i^2 = -1)

### Geometric Representation

Complex numbers can be visualized on the **complex plane**, where the x-axis represents the real part and the y-axis represents the imaginary part.

The **magnitude** or **modulus** of z is given by:

$$|z| = \sqrt{a^2 + b^2}$$

The **argument** (angle θ) is:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Thus, we can represent z in **polar form**:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

This is known as **Euler's formula**, a fundamental concept in quantum theory, as it elegantly connects complex numbers with rotations and phase shifts.

### Example:

Let $z_1 = 1 + i$ and $z_2 = 2 - i$

Then:

- Addition: $z_1 + z_2 = (1+2) + (1-1)i = 3$
- Multiplication:

$$z_1 z_2 = (1+i)(2-i) = 2 - i + 2i - i^2 = 3 + i$$

- Magnitude of $z_1$:

$$|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

- Polar form:

$$z_1 = \sqrt{2}e^{i\pi/4}$$

In quantum computing, such numbers are used to express **probability amplitudes**, where the square of the magnitude gives the probability of observing a particular quantum state.

**Quantum Connection:**

A **qubit** (quantum bit) is a unit vector represented as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha,\beta \in C$ (complex numbers), and:

$$|\alpha|^2 + |\beta|^2 = 1$$

This normalization ensures that the total probability of measuring the qubit in either state is 1.

## 2. Vector Spaces

A **vector space** (or **linear space**) is a mathematical structure formed by a collection of objects called **vectors**, which can be added together and multiplied (scaled) by numbers known as **scalars**.

In **quantum computing**, these vectors represent **quantum states**, and the scalars are **complex numbers**.

### Definition

A **vector space** V over a field C (complex numbers) is a set of elements (vectors) satisfying the following properties:

1. **Closure under addition:**
   If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$
2. **Closure under scalar multiplication:**
   If $\alpha \in \mathbb{C}$ and $\mathbf{v} \in V$, then $\alpha\mathbf{v} \in V$
3. **Existence of a zero vector:**
   There exists $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
4. **Existence of additive inverses:**
   For every $\mathbf{v} \in V$, there exists $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. **Distributive and associative properties** for both addition and scalar multiplication.

### Quantum Vector Space

In quantum computing, **each qubit** is represented as a **vector** in a **two-dimensional complex vector space** C^2.

The basis vectors are denoted as:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any single qubit can then be represented as a linear combination (superposition) of these two basis vectors:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where

$\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1.$

### Example 1:

Let's define a quantum state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

Then in matrix form:

$$|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$$

Here:

- The real and imaginary parts come from complex numbers.
- The vector space is **2-dimensional over complex numbers**.

### Example 2: Multi-Qubit Systems

For **two qubits**, the vector space becomes **4-dimensional**:

For **two qubits**, the vector space becomes **4-dimensional**:

$$\mathbb{C}^4 = \text{span}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

So, a general two-qubit state can be expressed as:

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$$

**Quantum Relevance:** The vector space framework allows:

- Representation of **superposition** (combination of states)
- Application of **linear operators** (quantum gates)
- Measurement as **projection** onto basis vectors

Quantum algorithms like **Grover's** and **Shor's** exploit these linear transformations to achieve exponential speedups over classical computation.

# 3. Matrices and Operators

In quantum computing, **matrices** and **operators** describe **transformations** applied to quantum states. While vectors represent **states**, matrices represent **actions** (operations) that **evolve** or **manipulate** those states.

These transformations correspond to **quantum gates**, the building blocks of quantum circuits.

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### 3.1 Matrices in Quantum Systems

A **matrix** is a rectangular array of numbers (usually complex numbers) arranged in rows and columns.

**For example:**

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

In quantum mechanics:

- Vectors (kets like $|\psi\rangle$) are **columns**.
- Operators (gates) are **matrices** that act on those columns.

The **result** of applying an operator A to a quantum state $|\psi\rangle$ is another state $|\phi\rangle$:

$$|\phi\rangle = A|\psi\rangle$$

### 3.2 Linear Operators

A **linear operator** A on a vector space V satisfies:

$$A(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha A|\psi\rangle + \beta A|\phi\rangle$$

for all scalars $\alpha,\beta \in C$ and vectors $|\psi\rangle,|\phi\rangle \in V$. These operators preserve the **structure of superposition**, which is crucial for quantum computation.

### 3.3 Unitary Operators

In quantum computing, **all evolution** of a closed quantum system is governed by **unitary operators**.

An operator U is **unitary** if:

$$U^\dagger U = UU^\dagger = I$$

where:

- U† is the **conjugate transpose** (Hermitian adjoint) of U
- I is the **identity matrix**

This property ensures **probability conservation** — the total probability of all possible outcomes remains 1 after any operation.

### Example: Common Quantum Gates as Matrices

| Gate | Symbol | Matrix Representation | Effect on Basis States |
|---|---|---|---|
| Identity | $I$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | Leaves state unchanged |
| Pauli-X (NOT) | $X$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | Swaps ( |
| Pauli-Y | $Y$ | $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ | Rotates state around Y-axis |
| Pauli-Z | $Z$ | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | Adds a phase to ( |
| Hadamard | $H$ | $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ | Creates superposition |
| Phase Gate | $S$ | $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ | Adds a phase of $\pi/2$ to ( |

## 3.4 Hermitian Operators

A matrix A is **Hermitian** if: $A^{\dagger}=A$

Hermitian operators correspond to **observables** (measurable physical quantities), such as energy, spin, or momentum.

For instance, in quantum mechanics, the **Hamiltonian** H (total energy operator) is Hermitian, ensuring that measurement outcomes are **real numbers**.

### 3.5 Example: Applying a Matrix Operator

**Let:**

$$|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

**Then:**

$$X|\psi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

Thus, applying the Pauli-X gate flips the qubit state from $|0\rangle$ to $|1\rangle$ .

# 4. Inner & Outer Product

In quantum computing, **inner** and **outer products** are vital linear algebraic operations that help define **probabilities**, **orthogonality**, and **state projection**.
They serve as the mathematical link between **quantum states** and **measurement outcomes**.

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### 4.1 Inner Product (Dot Product)

The **inner product** of two quantum states (vectors) $|\psi\rangle$ and $|\phi\rangle$ is a scalar **complex number** that measures how similar or "aligned" they are.

**Bra-Ket: Inner Product ($\langle\psi|\phi\rangle$) :**

For two qubit states $|\psi\rangle$ and $|\phi\rangle$, the inner product is given by:

$$\langle\psi|\phi\rangle = \alpha^*\gamma + \beta^*\delta$$

where:

- $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$
- $\alpha$, $\beta$, $\gamma$, and $\delta$ are complex numbers, and $\alpha^*$ and $\beta^*$ are the complex conjugates of $\alpha$ and $\beta$.

Suppose If:

$$|\psi\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

Then:

$$\langle\phi|\psi\rangle = b_1^* a_1 + b_2^* a_2$$

where * denotes the **complex conjugate**.

## Properties of Inner Product

For all $|\psi\rangle, |\phi\rangle, |\chi\rangle \in V$ and $\alpha \in \mathbb{C}$:

1. **Conjugate symmetry:**

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$$

2. **Linearity in the second argument:**

$$\langle\phi|(\alpha|\psi\rangle + |\chi\rangle) = \alpha\langle\phi|\psi\rangle + \langle\phi|\chi\rangle$$

3. **Positivity:**

$$\langle\psi|\psi\rangle \geq 0$$

   and $\langle\psi|\psi\rangle = 0$ only if $|\psi\rangle = 0$.

## Example: Inner Product of Quantum States

Let:

$$|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix}$$

Then:

$$\langle\phi|\psi\rangle = \left(\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}(1 - 1) = 0$$

Thus, $|\psi\rangle$ and $|\phi\rangle$are **orthogonal** — they represent mutually exclusive quantum states.

### Quantum Interpretation

The **square of the magnitude** of an inner product gives the **probability** of transitioning from one state to another:

$$P = |\langle \phi | \psi \rangle|^2$$

Orthogonal states ($\langle \phi | \psi \rangle = 0$) cannot transform into each other through measurement.

## Another Example for Inner Product

Consider a qubit in the following state:

$$|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$$

In ket notation:

$$|\psi\rangle = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

The bra for this state would be:

$$\langle \psi| = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

Let's say we have another qubit state $|\phi\rangle$ given by:

$$|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

In ket notation:

$$|\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

The bra for this state would be:

$$\langle \phi| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Now, let's calculate the **inner product** $\langle\phi|\psi\rangle$, which gives the overlap between the two states:

$$\langle\phi|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

Perform the multiplication:

$$\langle\phi|\psi\rangle = \frac{1}{\sqrt{2}} \times \frac{3}{5} + \frac{1}{\sqrt{2}} \times \frac{4}{5} = \frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}} = \frac{7}{5\sqrt{2}}$$

## 4.2 Outer Product

The **outer product** of two vectors creates an **operator** (matrix) from them. It is denoted by:

### Ket-Bra: Outer Product (|ψ⟩⟨ϕ|)

If:

$$|\psi\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \langle\phi| = \begin{bmatrix} b_1^* & b_2^* \end{bmatrix},$$

then:

$$|\psi\rangle\langle\phi| = \begin{bmatrix} a_1 b_1^* & a_1 b_2^* \\ a_2 b_1^* & a_2 b_2^* \end{bmatrix}$$

### Example: Outer Product as Projection Operator

Let:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \langle 0| = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Then:

$$|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

This acts as a **projector** onto the state $|0\rangle$. Similarly, $|1\rangle\langle 1|$ projects onto the $|1\rangle$ state.

## Quantum Application

- Outer products are used to **construct quantum operators** such as **density matrices** and **measurement projectors**.
- For instance, the density matrix of a pure state $|\psi\rangle$ is:

  $\rho=|\psi\rangle\langle\psi|$

This operator contains complete information about the quantum system.

## Another Example for Outer Product

The **outer product** $|\psi\rangle\langle\phi|$ results in a matrix (operator):

$$|\psi\rangle\langle\phi| = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Perform the matrix multiplication:

$$|\psi\rangle\langle\phi| = \begin{pmatrix} \frac{3}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{4}{5\sqrt{2}} \end{pmatrix}$$

## Example: Density Matrix

If:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Then:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This density matrix describes a **superposition state** with equal probabilities for $|0\rangle$ and $|1\rangle$.

## 4.3 Significance in Quantum Computing

1. **Measurement Operators:**
   Inner products determine measurement probabilities, e.g.,
   $$P(|\phi\rangle) = |\langle\phi|\psi\rangle|^2$$
2. **State Projection:**
   Outer products help represent projection operations like
   $$P_0 = |0\rangle\langle 0|, \quad P_1 = |1\rangle\langle 1|$$
3. **Density Matrices:**
   For a pure state $|\psi\rangle$, the density matrix is given by
   $$\rho = |\psi\rangle\langle\psi|$$
   which encodes all measurable information about the quantum system.

## 5. Magnitude and Normalization

In Quantum Computing, every quantum state (or qubit) is represented as a **vector** in a complex vector space. To ensure that a qubit represents a valid physical state, it must be **normalized** — meaning its total probability equals **1**.

Normalization and magnitude play a central role in determining the physical validity of a quantum state.

### 5.1 Magnitude of a Quantum State

The **magnitude** (or **norm**) of a quantum state |ψ⟩ is defined as the square root of the inner product of the state with itself:

$$||\psi|| = \sqrt{\langle\psi|\psi\rangle}$$

If

$$|\psi\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

Then

$$||\psi|| = \sqrt{|\alpha|^2 + |\beta|^2}$$

This norm represents the total probability amplitude of the state vector.

## 5.2 Normalization Condition

A **normalized** quantum state satisfies the following condition:

$$\langle\psi|\psi\rangle = 1$$

That is, the sum of the squared magnitudes of all amplitudes equals one.

$$|\alpha|^2 + |\beta|^2 = 1$$

This ensures that when a measurement is made, the total probability of all possible outcomes (e.g., $|0\rangle$ and $|1\rangle$) is exactly 1.

**Example:**

Let's say the qubit vector is:

$$|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

The magnitude would be:

$$||\psi|| = \sqrt{\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$



### *Example 1: Normalized Qubit*

Let

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then:

$$\langle\psi|\psi\rangle = \frac{1}{2}(1^2 + 1^2) = 1$$

Hence, this qubit is **normalized**.

### *Example 2: Non-normalized Qubit*

Let

$$|\phi\rangle = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Then:

$$\langle\phi|\phi\rangle = 2^2 + 1^2 = 5$$

To normalize |ϕ⟩, divide by its magnitude:

$$||\phi|| = \sqrt{5}, \quad |\phi_{normalized}\rangle = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Now,

$$||\phi|| = \sqrt{5}, \quad |\phi_{normalized}\rangle = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Now,

$$\langle\phi_{normalized}|\phi_{normalized}\rangle = 1$$

### 5.3 Physical Meaning

In a quantum system, normalization ensures **probability conservation**. If a qubit is in state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

then:

- The probability of measuring $|0\rangle$ is $|\alpha|^2$
- The probability of measuring $|1\rangle$ is $|\beta|^2$
- Total probability = $|\alpha|^2 + |\beta|^2 = 1$

This property guarantees that one of the outcomes **must occur** upon measurement.

### 5.4 Geometric Interpretation

Normalization can be visualized geometrically.
A normalized qubit lies on the **surface of the unit sphere** (known as the **Bloch Sphere**) in complex 3D space.
Thus, every valid qubit corresponds to a point on this sphere, ensuring it has a magnitude of 1.

**Normalization Process**

To normalize $|\psi\rangle$, you divide each component by the norm of the vector:

$$|\psi_{\text{normalized}}\rangle = \frac{1}{||\psi||}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{||\psi||} \\ \frac{b}{||\psi||} \end{pmatrix}$$

**Example**

Let's consider a qubit vector:

$$|\psi\rangle = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

**Step 1: Calculate the norm of $|\psi\rangle$:**

$$\|\psi\| = \sqrt{|3|^2 + |4|^2} = \sqrt{9+16} = \sqrt{25} = 5$$

**Step 2: Normalize the vector:**

To normalize $|\psi\rangle$, divide each component by the norm:

$$|\psi_{\text{normalized}}\rangle = \frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix}$$

So the normalized qubit vector is:

$$|\psi_{\text{normalized}}\rangle = \begin{pmatrix}0.6\\0.8\end{pmatrix}$$

**Step 3: Verify the normalization:**

Let's check the magnitude of the normalized vector:

$$\|\psi_{\text{normalized}}\| = \sqrt{(0.6)^2 + (0.8)^2} = \sqrt{0.36+0.64} = \sqrt{1} = 1$$

Since the magnitude is 1, the vector is properly normalized.

Normalization ensures that the qubit vector has a magnitude of 1, making it consistent with the principles of quantum mechanics. In this example, the vector $\begin{pmatrix}3\\4\end{pmatrix}$ was normalized to $\begin{pmatrix}0.6\\0.8\end{pmatrix}$.

## Perpendicular and Parallel Qubit Vectors

Two qubit vectors are perpendicular if their inner product (dot product) is zero. For example:

- **Qubit Vector 1:** $|\phi_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$
- **Qubit Vector 2:** $|\phi_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\langle\phi_1|\phi_2\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = (1\cdot 0) + (0\cdot 1) = 0$$

Since the inner product is zero, $|\phi_1\rangle$ and $|\phi_2\rangle$ are perpendicular.

## Parallel Qubit Vectors

Two vectors are parallel if one is a scalar multiple of the other. Let's check if $|\psi_2\rangle$ is a scalar multiple of $|\psi_1\rangle$.

**Example:**

- **Qubit Vector 1:** $|\psi_1\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- **Qubit Vector 2:** $|\psi_2\rangle = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

We can express $|\psi_2\rangle$ as:

$$|\psi_2\rangle = 2 \cdot |\psi_1\rangle = 2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Since $|\psi_2\rangle$ is exactly $2$ times $|\psi_1\rangle$, the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ are parallel.

## 6. Angle Between Quantum Vectors

In Quantum Computing, the **angle** between two quantum state vectors represents their **degree of similarity** or **overlap**. The closer two states are (smaller angle), the more similar they are in terms of probability amplitudes. Orthogonal states (angle = 90°) represent **mutually exclusive** or **distinguishable** quantum outcomes.

### 6.1 Definition

For two quantum states $|\psi\rangle$ and $|\phi\rangle$, the angle $\theta$ between them is defined as:

$$\cos\theta = \frac{|\langle\psi|\phi\rangle|}{||\psi||\ ||\phi||}$$

If both states are normalized (i.e., $||\psi||=||\phi||=1$), then:

$$\cos\theta = |\langle\psi|\phi\rangle|$$

This relationship directly connects the **inner product** with the **geometric interpretation** of the quantum state space.

### 6.2 Interpretation

- When $\theta = 0^\circ$:

  The states are **identical** — $|\psi\rangle = |\phi\rangle$.

  $$|\langle\psi|\phi\rangle| = 1$$

- When $\theta = 90^\circ$:

  The states are **orthogonal** — completely distinct.

  $$|\langle\psi|\phi\rangle| = 0$$

- When $0^\circ < \theta < 90^\circ$:

  The states are **partially similar** — they have some probability overlap.

### Example:

Let's consider two qubit vectors:

$$|\psi_1\rangle = \begin{pmatrix} 1+i \\ 0 \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

**Step 1: Compute the Inner Product**

$$\langle\psi_1|\psi_2\rangle = (1-i)\cdot 0 + 0\cdot 1 = 0$$

The inner product is $0$.

**Step 2: Find the Magnitudes**

$$\|\psi_1\| = \sqrt{|1+i|^2 + 0^2} = \sqrt{(1^2+1^2)} = \sqrt{2}$$

$$\|\psi_2\| = \sqrt{0^2 + |1|^2} = \sqrt{1} = 1$$

**Step 3: Calculate the Cosine of the Angle**

Since the inner product is $0$, $\cos(\theta)$ will also be $0$:

$$\cos(\theta) = \frac{0}{\sqrt{2} \cdot 1} = 0$$

**Step 4: Find the Angle**

$$\theta = \arccos(0) = \frac{\pi}{2} \text{ radians} = 90^\circ$$

The angle $\theta$ between the two qubit vectors $|\psi_1\rangle = \begin{pmatrix} 1+i \\ 0 \end{pmatrix}$ and $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $90^\circ$, meaning they are orthogonal (perpendicular) to each other.

## 6.3 Example 1: Orthogonal States

Let

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then,

$$\langle 0|1\rangle = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90^\circ$$

Hence, $|0\rangle$ and $|1\rangle$ are **orthogonal** — they represent **mutually exclusive measurement outcomes**.

### 6.4 Example 2: Non-Orthogonal States

Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

Then,

$$\langle\phi|\psi\rangle = \frac{1}{\sqrt{2}} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ$$

Thus, the states are **partially overlapping** — not orthogonal, but not identical either.

## 6.5 Geometric Visualization (Bloch Sphere)

On the **Bloch Sphere**, the angle between two quantum states corresponds to the **geodesic distance** between their points on the sphere.

For qubit states:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Here:

- $\theta$ → polar angle (controls superposition amplitude)
- $\phi$ → azimuthal phase (controls relative phase difference)

Thus, the **angle between vectors** encodes **quantum distinguishability** — how easily one state can be told apart from another upon measurement.

### 6.6 Importance in Quantum Computing

1. **Quantum State Fidelity:**
   The fidelity between two states is defined as

$$F(|\psi\rangle, |\phi\rangle) = |\langle\psi|\phi\rangle|^2 = \cos^2\theta$$

It measures how “close” two quantum states are.

- **Error Detection:**

Orthogonal states ensure perfect distinguishability — critical for **quantum error correction**.

- **Quantum Gates:**

Operations like the **Hadamard gate** create specific angular separations (e.g., 45° rotations) between basis states, forming **superpositions**

## 7. Linear Combination of Qubit Vectors

A **linear combination** of qubit vectors is a fundamental operation in quantum computing and linear algebra. It allows us to describe **superpositions** — one of the most essential and unique features of quantum mechanics.

A **linear combination** of two qubit vectors involves creating a new qubit vector by adding the vectors together, each multiplied by a scalar (which can be a complex number).

Given two qubit vectors $|\psi_1\rangle$ and $|\psi_2\rangle$, a linear combination of these vectors can be expressed as:

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$$

where $c_1$ and $c_2$ are complex numbers (scalars).

**Example :**

Consider two qubit vectors:

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These are the basis states $|0\rangle$ and $|1\rangle$, respectively.

Let's form a linear combination:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

In essence, **any valid quantum state** can be formed by linearly combining basis states such as $|0\rangle$ and $|1\rangle$.

## 7.1 Definition

If $|0\rangle$ and $|1\rangle$ are the standard **computational basis states**, then any single-qubit state $|\psi\rangle$ can be written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where:

- $\alpha$ and $\beta$ are **complex probability amplitudes**, and
- the normalization condition holds:

$$|\alpha|^2 + |\beta|^2 = 1$$

Thus, $|\psi\rangle$ is a **linear combination** of $|0\rangle$ and $|1\rangle$.

### 7.2 Example 1: Equal Superposition

In general, the superposition state can have any complex coefficients $c_1$ and $c_2$:

$$|\psi\rangle = c_1|0\rangle + c_2|1\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The state is normalized if:

$$|c_1|^2 + |c_2|^2 = 1$$

**Consider**

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

This represents an equal probability of being in both $|0\rangle$ and $|1\rangle$ states.

Probabilities:

$$P(|0\rangle) = P(|1\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Such a state is often generated using the **Hadamard gate (H)**:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

### 7.3 Example 2: Unequal Superposition

$$|\phi\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$$

Probabilities:

$$P(|0\rangle) = \frac{3}{4}, \quad P(|1\rangle) = \frac{1}{4}$$

Here, the qubit is more likely to collapse to |0⟩ upon measurement.

**Example : Biased Towards |0⟩**

In this state, the probability of measuring $|0\rangle$ is higher than that of measuring $|1\rangle$:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

**Probabilities:**

- $P(0) = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4} = 75\%$
- $P(1) = \left|\frac{1}{2}\right|^2 = \frac{1}{4} = 25\%$

Example : **Biased Towards |1⟩**

Here, the probability of measuring $|1\rangle$ is higher:

$$|\psi\rangle = \frac{1}{3}|0\rangle + \frac{2\sqrt{2}}{3}|1\rangle$$

**Probabilities:**

- $P(0) = \left|\frac{1}{3}\right|^2 = \frac{1}{9} \approx 11.11\%$
- $P(1) = \left|\frac{2\sqrt{2}}{3}\right|^2 = \frac{8}{9} \approx 88.89\%$

Example : **Closer to |0⟩**

This state has a higher probability of being in $|0\rangle$ but still a significant chance of being in $|1\rangle$:

$$|\psi\rangle = \frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle$$

**Probabilities:**

- $P(0) = \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5} = 80\%$
- $P(1) = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5} = 20\%$

Example : **Very Close to |1⟩**

This state has a high probability of being in $|1\rangle$:

$$|\psi\rangle = \frac{1}{4}|0\rangle + \frac{\sqrt{15}}{4}|1\rangle$$

**Probabilities:**

- $P(0) = \left|\frac{1}{4}\right|^2 = \frac{1}{16} = 6.25\%$
- $P(1) = \left|\frac{\sqrt{15}}{4}\right|^2 = \frac{15}{16} = 93.75\%$

**Note:**

- **Ket notation $|\psi\rangle$**: Represents the state as a **column vector**.
- **Bra notation $\langle\psi|$**: Represents the conjugate transpose of the ket as a **row vector.**
- **Inner product $\langle\phi|\psi\rangle$**: Gives a **scalar**, indicating the **overlap** between two quantum states.
- **Outer product $|\psi\rangle\langle\phi|$**: Results in a **matrix**, useful for constructing quantum operators.

### 7.4 Linear Independence and Span

The states $|0\rangle$ and $|1\rangle$ are **linearly independent**. This means no scalar multiple of one can produce the other:

$a|0\rangle + b|1\rangle = 0 \implies a = b = 0$

The set $\{|0\rangle, |1\rangle\}$ thus **spans** the two-dimensional qubit space, meaning **any qubit state** can be represented as their linear combination.

### 7.5 Linear Combination in Multi-Qubit Systems

For two qubits, the basis expands as:

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

A general two-qubit state is a linear combination of these four basis states:

$$|\Psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

Normalization condition:

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

### 7.6 Importance in Quantum Computing

1. **Superposition Principle:** Linear combination forms the mathematical foundation of **quantum superposition**, enabling quantum parallelism.
2. **Quantum Algorithms:** Algorithms like **Grover's** and **Shor's** rely on manipulating linear combinations of states to amplify or suppress certain outcomes.
3. **Quantum Interference:** When multiple quantum states combine, their amplitudes **interfere** (constructively or destructively), directly resulting from their linear combination properties.

### 7.7 Geometric Interpretation

On the **Bloch Sphere**, every possible linear combination of $|0\rangle$ and $|1\rangle$ corresponds to a unique point on the surface.
The coefficients $\alpha$ and $\beta$ determine:

- The **latitude** (magnitude ratio)
- The **longitude** (phase difference)

This gives a clear geometric representation of how **linear combination = superposition = rotation on the Bloch sphere**.

## 8. **Hilbert Space in Quantum Computing**

A **Hilbert space** in quantum computing is a mathematical framework used to describe the state space of quantum systems. It is a complete inner product space where:

- **Vectors** represent quantum states.
- **Inner product** defines the overlap or similarity between states.
- **Norm** of a vector represents the probability amplitude of finding the system in that state.
- **Unitary Operators** represent quantum gates.
- **Projection Operators** represent measurements.
- **Probabilities** are calculated based on the norms and inner products.

In quantum computing, the Hilbert space $\mathbb{C}^2$ for a single qubit includes:

- **Basis Vectors**: $|0\rangle$ and $|1\rangle$, represented as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.
- **State Vectors**: Any qubit state can be expressed as a superposition of the basis vectors.
- **Unitary Operators**: Transform the state vectors; for example, the Hadamard gate.
- **Measurement**: Projects the state vector onto the basis vectors and gives probabilities for measurement outcomes.

## Basis in Quantum Mechanics

- In **quantum mechanics**, a basis typically refers to a set of orthonormal vectors in a **Hilbert space.**
- For qubits, the basis vectors are often represented as **|0⟩ and |1⟩,** which are the standard basis vectors for a single qubit

### |0> and |1> are ortho normal basis

## 1. Orthogonality

To check orthogonality, we calculate the inner product (dot product) of $|0\rangle$ and $|1\rangle$:

$$\langle 0|1\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1 \times 0) + (0 \times 1) = 0$$

Since $\langle 0|1\rangle = 0$, the vectors $|0\rangle$ and $|1\rangle$ are orthogonal.

### 2. Normalization

To check normalization, we calculate the norm of each vector:

- For $|0\rangle$:

$$\| \, |0\rangle \, \| = \sqrt{\langle 0|0\rangle} = \sqrt{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \sqrt{(1 \times 1) + (0 \times 0)} = \sqrt{1} = 1$$

- For $|1\rangle$:

$$\| \, |1\rangle \, \| = \sqrt{\langle 1|1\rangle} = \sqrt{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \sqrt{(0 \times 0) + (1 \times 1)} = \sqrt{1} = 1$$

Both $|0\rangle$ and $|1\rangle$ are normalized since their norms are equal to 1.

## 9. Tensor Products

- If we have two vector spaces V and W, their tensor product of V and W is a new vector space formed from all possible combinations of vectors from V and W.
- The dimension of the tensor product space is the product of the dimensions of the individual spaces.
- For example, if V has dimension m and W has dimension n, then tensor product of V and W has dimension m×n.

### Tensor Product Notation

The tensor product of two vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ is denoted as:

$$|\psi_1\rangle \otimes |\psi_2\rangle$$

It is often written simply as $|\psi_1\rangle|\psi_2\rangle$ or $|\psi_1\psi_2\rangle$.

**Example 1 :**

$$X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0(I) & 1(I) \\ 1(I) & 0(I) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

**Example 2:**

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ 1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

## Example 3:

Consider two qubits in the states $|\psi_1\rangle$ and $|\psi_2\rangle$:

- Let $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$
- Let $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

The tensor product $|\psi_1\rangle \otimes |\psi_2\rangle$ is:

$$|\psi_1\rangle \otimes |\psi_2\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$$

Expanding this:

$$|\psi_1\rangle \otimes |\psi_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

## Example 4:

Let's take specific qubit states:

- $|\psi_1\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $|\psi_2\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The tensor product is:

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This results in:

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \cdot 0 \\ 1 \cdot 1 \\ 0 \cdot 0 \\ 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

**Example 5:**

Now, let's consider two qubits each in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$:

- $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

The tensor product gives:

$$|\psi_1\rangle \otimes |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \frac{1}{\sqrt{2} \cdot \sqrt{2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$